Synchronization and its slow decay in noisy oscillators with simplicial interactions

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Previous studies on oscillator populations with two-simplex interaction have reported novel phenomena such as discontinuous desynchronization transitions and multistability of synchronized states. However, the noise effect is not well understood. Here, we study a higher-order network of noisy oscillators with generic interactions consisting of one-simplex and two types of two-simplex interactions. We observe that when a type of two-simplex interaction is dominant, synchrony is eroded and eventually disappears even for infinitesimally weak noise. Nevertheless, synchronized states may persist for extended periods, with the lifetime increasing approximately exponentially with the strength of the two-simplex interaction. When one-simplex or another type of two-simplex interaction is sufficiently strong, noise erosion is prevented, and synchronized states become persistent. A weakly nonlinear analysis reveals that as one-simplex coupling increases, the synchronized state appears supercritically or subscritically, depending on the interaction strength. Furthermore, assuming weak noise and using Kramers' rate theory, we derive a closed dynamical equation for the Kuramoto order parameter, from which the time scale of the erosion process is derived. Our study elucidates the synchronization and desynchronization of oscillator assemblies in higher-order networks and is expected to provide insights into such systems' design and control principles.

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I. INTRODUCTION

Synchronization is a major field of study in not only physics but also chemistry, biology, engineering, and sociology [1-5]. Examples include pacemaker cells in the heart [6], laser arrays [7], applauding audiences [8,9], power grids consisting of alternating current (ac) generators [10], and Josephson junctions [11,12]. In addition to synchronization, understanding desynchronization in oscillator assemblies is crucial. For example, while synchronization of circadian pacemaker cells in the brain is essential for mammals to maintain 24-h activity rhythm, their transient desynchronization, triggered by a phase shift of light-dark cycles, is a putative cause of jet lag symptoms [13,14]. Desynchronization is also significant in neurological disorders such as Parkinson's disease [15] and epilepsy [16]. Methods for promoting desynchronization in this context have been actively studied both theoretically and experimentally [17].

Phase oscillator models, including Kuramoto's model [18], are widely recognized for their utility, not only in understanding synchronization [2,19] but also in controlling real-world systems [20,21]. While the classical Kuramoto model considers pairwise interactions between oscillators, recent studies have extended the model to allow for nonpairwise interactions [22–25]. Such structures are often called simplexes, where *n*-simplex describes an interaction between n + 1 oscillators [26]. Research on brain dynamics [27,28] or social

been conducted recently [30–34] In contrast, for noisy phase oscillators, Komarov and Pikovsky [35] studied a system

an important role in such systems.

with a particular type of two-simplicial coupling alone and reported that no stable synchronized states exist in the limit of an infinite number of oscillators. They mainly focused on the synchronized states that exist only in small populations. Desynchronization is expected in a large population, and as mentioned above, clarifying its process is essential.

phenomena [29] has suggested that simplicial structures play

tions, Tanaka and Aoyagi [22] noted that multiple stable

synchronized states appear as two clusters with different pop-

ulation ratios. Moreover, Skardal and Arenas showed that

abrupt desynchronization transitions occur as the interaction

strength decreases [23]. Skardal and Arenas reported that two-

and three-simplicial couplings promote abrupt synchroniza-

tion transitions in the presence of one-simplex interactions

[24]. In addition, numerous relevant studies on the synchro-

nization of noiseless oscillators in higher-order networks have

In noiseless phase oscillators with two-simplex interac-

In the present study, we consider a large population of noisy phase oscillators with one-simplex and two types of two-simplex couplings. Although steady synchronized states do not exist when a type of two-simplex interaction, denoted as type-a coupling, is dominant, we demonstrate that the population is transiently synchronized for an extended period and then abruptly desynchronized. Assuming weak noise and exploiting Kramers' rate theory, we derive a closed dynamical equation for the Kuramoto order parameter by which the desynchronization process is reproduced and the exponential dependence of the lifetime of the synchronized states

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ical strength of type-a coupling.

II. MODEL AND SIMULATION RESULTS

transition changes from continuous to discontinuous at a crit-

We consider a system of identical phase oscillators subjected to independent noise and globally coupled with one-simplex (i.e., two-body) and two types of two-simplex (i.e., three-body) interactions, given as

$$\dot{\theta}_m = \omega_m + \frac{K_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_m) + \frac{1}{N^2} \sum_{j,k=1}^N [K_{2a} \sin(\theta_j + \theta_k - 2\theta_m) + K_{2b} \sin(2\theta_j - \theta_k - \theta_m)] + \xi_m(t), \qquad (1)$$

where ω_m and θ_m are the intrinsic frequency and the phase of the oscillator m ($1 \le m \le N$), respectively. The term $\xi_m(t)$ represents Gaussian white noise with zero mean, δ -correlated in time and independent for different oscillators. Specifically, $\langle \xi_m(t) \rangle = 0$, $\langle \xi_m(t) \xi_n(\tau) \rangle = 2D\delta_{mn}\delta(t-\tau)$, where $D \ge 0$ is the noise strength. The remaining terms describe interactions, where $K_1 \ge 0$, $K_{2a} \ge 0$, and $K_{2b} \ge 0$ are the coupling strengths of one-simplex, type-a two-simplex, and type-b twosimplex interactions, respectively. This model is motivated by the fact that two types of two-simplicial interaction terms emerge from higher-order phase reductions [36,37], which can be considered a natural extension of the previously proposed models [23,24,35].

Next, we introduce Kuramoto-Daido order parameters [38,39], defined as

$$Z_{l}(t) = R_{l}(t)e^{i\Theta_{l}(t)} =: \frac{1}{N}\sum_{j=1}^{N}e^{il\theta_{j}(t)}, \quad \text{for } l = \pm 1, \pm 2, \dots,$$
(2)

where $R_l \in [0, 1]$ and $\Theta_l \in (-\pi, \pi]$ represent the amplitudes and the mean phases, respectively. Note that Z_{-l} is the complex conjugate of Z_l . We utilize this relationship in Sec. III B, where we perform the stability analysis of the incoherent state. Note that R_1 assumes 1 and 0 for the in-phase state (i.e., $\theta_j = \theta_0$ for $1 \le j \le N$) and the fully desynchronized state (i.e., θ_j is uniformly distributed within $(-\pi, \pi]$), respectively. Similarly, $R_2 = 1$ for the two-cluster states given by $\theta_j = \theta_0$ or $\theta_0 + \pi$ for $1 \le j \le N$ and $R_2 = 0$ for the fully desynchronized state. Using R_l and Θ_l , Eq. (1) may be rewritten as

$$\dot{\theta}_m = \omega_m + K_1 R_1 \sin(\Theta_1 - \theta_m) + K_{2a} R_1^2 \sin(2\Theta_1 - 2\theta_m) + K_{2b} R_1 R_2 \sin(\Theta_2 - \Theta_1 - \theta_m) + \xi_m(t).$$
(3)

We can observe that the type-a interaction, given as the third term on the right-hand side, tends to make θ_m either Θ_1 or

 $\Theta_1 + \pi$. Therefore, one can suspect that the type-a interaction is likely to promote the formation of two-cluster states, which is indeed observed when D = 0 [23]. Based on this observation, we numerically investigate the dynamics for the initial condition of two-cluster states. Specifically, we set $\theta_m(0) = 0$ and π for $1 \le m \le \eta N$ and otherwise, respectively, where $\eta \ge \frac{1}{2}$ is the initial population ratio of the two clusters. Note that $\eta = 1$ corresponds to the one-cluster state, i.e., in-phase synchrony.

We first consider the case of identical oscillators, i.e., $\omega_m = \omega_0$ for $1 \leq m \leq N$. Without loss of generality, we set $\omega_0 = 0$. Figure 1(a) illustrates the time evolution of the order parameter R_1 . For $K_1 = 0.1$, $K_{2b} = 0$, and D = 0.0 (black solid line), we observe that R_1 is almost constant, indicating that the two-cluster state is stable. However, in the presence of noise, we observe qualitatively different behaviors. For $K_1 = 0.1$ and D = 0.1 (orange solid line), R_1 slowly decreases and abruptly vanishes. Thus, in the presence of noise, the two-cluster state is actually not stable but meta-stable with a long lifetime. We refer to this phenomenon as the noise erosion of the synchronized state because of its slow process. The evolution of the phase distribution during this process is shown in Fig. 2(a). We qualitatively obtain the same results for small or vanishing K_1 values. In contrast, for $K_1 = 0.3$, $K_{2b} =$ 0, and D = 0.1 (blue solid line), R_1 seems to approach a particular nonvanishing value. For this parameter set, we also test the initial condition of the fully desynchronized state (dotted green line) and observe the evolution of the phase distributions [Fig. 2(b)], suggesting that the system approaches a particular two-cluster state independent of the initial condition. When the type-b interaction is present (i.e., $K_{2b} > 0$) instead of the one-simplex interaction, similar results are obtained for the synchronized initial conditions, as shown in Fig. 1(b). However, for the desynchronized initial condition, R_1 is vanishingly small for all t > 0, indicating that the desynchronized state is stable. This behavior is preserved for larger K_{2b} values (the result is not shown herein).

Next, we consider the case of nonidentical oscillators. Specifically, the natural frequencies ω_m are drawn from a Lorentzian distribution $g(\omega)$ with mean ω_0 and width γ , i.e., $g(\omega) = \frac{\gamma}{\pi[(\omega - \omega_0)^2 + \gamma^2]}$. As shown in Figs. 1(c) and 1(d), we observe qualitatively similar features. Therefore, we expect that the cases of identical oscillators capture the essential properties of the system, and henceforth, we focus on the simpler case $\omega_m = \omega_0$ for ease of analysis.

III. ANALYSIS FOR STATIONARY STATE

A. Stationary distribution

We now establish a theory for understanding the synchronization and desynchronization processes. We consider a continuum limit $N \rightarrow \infty$ for analytical tractability. Specifically, the number of oscillators with a phase within $(\theta, \theta + d\theta)$ is described by $NP(\theta, t)d\theta$, where $P(\theta, t)$ is a probability density function. We start with the self-consistency approach to identify the steady states of the system. The order parameters are redefined as

$$Z_l(t) = \int_{-\pi}^{\pi} \exp(il\theta) P(\theta, t) d\theta.$$
(4)



FIG. 1. Time series of the order parameter R_1 for (a) $K_1 > 0$, $K_{2a} = 3.0$, $K_{2b} = 0$ and (b) $K_1 = 0$, $K_{2a} = 3.0$, $K_{2b} > 0$. We employ the two-cluster state with $\eta = 0.75$ for solid curves and the desynchronized state for dashed curves as the initial condition. [(c), (d)] Time series of the order parameter R_1 for (c) $K_1 > 0$, $K_{2b} = 0$ and (d) $K_1 = 0$, $K_{2b} > 0$ in a population of nonidentical oscillators. Type-a coupling strength is $K_{2a} = 3.0$. We employ the two-cluster state with $\eta = 0.75$ for solid curves and the desynchronized state for dashed curves as the initial condition. $N = 10^4$.

The Fokker-Planck equation equivalent to the Langevin equation (3) is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \theta} \left[K_1 R_1 \sin(\theta - \Theta_1) + K_{2a} R_1^2 \sin(2\theta - 2\Theta_1) + K_{2b} R_1 R_2 \sin(\theta + \Theta_1 - \Theta_2) \right] P + D \frac{\partial^2 P}{\partial \theta^2}.$$
(5)

As shown in Fig. 7 in Appendix A, we numerically observe that Θ_1 and Θ_2 evolve slowly, suggesting that they are constant in the limit $N \to \infty$. Moreover, $\Theta_2 = 2\Theta_1$ holds true for $t \ge 0$. We thus assume $\Theta_1 = \Theta_2 = 0$. Substituting $\Theta_1 = \Theta_2 = 0$ into Eq. (5), we obtain

$$\frac{\partial P(\theta, t)}{\partial t} = \frac{\partial}{\partial \theta} \Big[\big\{ K_1 R_1 \sin \theta + K_{2a} R_1^2 \sin 2\theta + K_{2b} R_1 R_2 \sin \theta \big\} P \Big] + D \frac{\partial^2 P}{\partial \theta^2}.$$
(6)

The stationary distribution $P_s(\theta)$ is found as a solution to $\partial_t P = 0$. The general solution is

$$P_{s}(\theta) = \exp\left(\frac{2K_{1}R_{1}\cos\theta + K_{2a}R_{1}^{2}\cos 2\theta + 2K_{2b}R_{1}R_{2}\sin\theta}{2D}\right) \cdot \\ \times \left[c_{1} + c_{2}\int_{-\pi}^{\theta}\exp\left(-\frac{2K_{1}R_{1}\cos y + K_{2a}R_{1}^{2}\cos 2y + 2K_{2b}R_{1}R_{2}\sin y}{2D}\right)dy\right],$$
(7)

where c_1 and c_2 are constants. Because $P_s(\theta) = P_s(\theta + 2\pi)$ for any θ , c_2 vanishes. Thus, we obtain the stationary distribution P_s as

$$P_{\rm s}(\theta) = c_1 \exp\left(\frac{2K_1R_{1\rm s}\cos\theta + K_{2\rm a}R_{1\rm s}^2\cos2\theta + 2K_{2\rm b}R_{1\rm s}R_{2\rm s}\cos\theta}{2D}\right),\tag{8}$$



FIG. 2. Time courses of the phase distribution $P(\theta, t)$. (a) $K_1 = 0.1$, $K_{2a} = 4.0$, $K_{2b} = 0$ and the initial condition is the two-cluster state with $\eta = 0.75$. (b) $K_1 = 0.3$, $K_{2a} = 4.0$, $K_{2b} = 0$ and the initial condition is the desynchronized state. We obtained the results from simulations of Eq. (3) with $N = 10^4$ oscillators.

where c_1 is the normalizing constant given as

Substituting Eq. (8) into Eq. (4), we obtain a set of selfconsistent equations for R_{ls} (l = 1, 2), given as

(10)

$$c_{1} = \frac{1}{\int_{-\pi}^{\pi} \exp\left(\frac{2K_{1}R_{1}\cos y + K_{2a}R_{1}^{2}\cos 2y + 2K_{2b}R_{1}R_{2}\cos y}{2D}\right) \mathrm{d}y}.$$
 (9)
$$R_{ls} = \int_{-\pi}^{\pi} P_{s}(\theta)\cos l\theta \,\mathrm{d}\theta =: S_{l}(R_{1s}, R_{2s}).$$



FIG. 3. Phase diagrams in (a) the (K_1, K_{2a}) plane for $K_{2b} = 0$ and (b) the (K_{2b}, K_{2a}) plane for $K_1 = 0$. The color scale describes the R_1 value of the stable synchronized state. The hatched region denotes the region where the desynchronized state is stable. In (a), the vertical line at $K_{2a} = 0.2$ denotes the critical coupling strength above which the desynchronized state is unstable. The solid and dashed lines denote the supercritical and subcritical bifurcation curves, respectively. The hatched colored region denotes the bistable region where both synchronized and desynchronized states are stable, and the dot denotes the bifurcation type switch at $K_1 = 2D$, as described by Eq. (48). [(c), (d)] Bifurcation diagrams for (c) $K_{2a} = 0.15$ and (d) $K_{2a} = 1.0$, where we fix $K_{2b} = 0$. In all the panels, we fix D = 0.1.

We observe that the right-hand side of Eq. (10) for l = 1 vanishes for $K_1 = 0$ and $K_{2b} = 0$ because then $P_s(\theta)$ is π -periodic. This implies that for $K_1 = 0$ and $K_{2b} = 0$, there is no stationary distribution except that corresponding to $R_{1s} = 0$, which is the uniform distribution $P_s(\theta) = \frac{1}{2\pi}$ [35]. In this case, R_2 also vanishes. Nontrivial steady distributions may only arise when $K_1 > 0$ or $K_{2b} > 0$.

B. Bifurcation analysis

We investigate the bifurcation of the system by numerically solving Eq. (10). First, we analyze the case of $K_{2b} = 0$. Here, we only need to consider Eq. (10) for l = 1 and numerically identifying R_{1s} is straightforward (see Appendix B). In Fig. 3(a), the phase diagram in the (K_1, K_{2a}) plane with $K_{2b} = 0$ is displayed, suggesting that a bifurcation occurs at $K_1 = K_c = 0.2$ for all K_{2a} . In Figs. 3(c) and 3(d), we plot R_{1s} as a function of K_1 for $K_{2a} = 0.15$ and $K_{2a} = 3.0$, respectively. We observe that supercritical and subcritical pitchfork bifurcations occur in Figs. 3(c) and 3(d), respectively. Moreover, the bifurcation type appears to change from supercritical to subcritical at $K_1 = K_c$, yielding the bistable region where both desynchronized and synchronized states are stable for $K_1 > K_c$. Note that $K_c = 0.2$ is equal to 2D. Next, we consider $K_{2b} > 0$. By numerically solving Eq. (10) for l = 1, 2 (see Appendix **B**), we obtain the R_{1s} value of the synchronized state for given K_{2a} and K_{2b} values, which is depicted in Fig. 3(b).

To elucidate the bifurcation structure, we perform a weakly nonlinear analysis by applying a previously proposed methodology [40] to our model. This analysis is carried out around the incoherent solution $Z_l = R_l = 0$ and is valid near the bifurcation point at which the solution becomes unstable. We will introduce a bifurcation parameter, which is treated as the sole small parameter in our analysis. Although the value of ω may arbitrarily be chosen and was set to 0 because of the translational symmetry in our model, we here assume $\omega = O(1)$ to apply the methodology proposed in Ref. [40]. This methodology can even apply to models where coupling terms break the translational symmetry, an advantage for future extensions.

Specifically, we expand the distribution $P(\theta, t)$ in series with respect to the order parameter Z_l as

$$P(\theta, t) = \frac{1}{2\pi} \left(1 + \sum_{l \neq 0} Z_{-l}(t) e^{il\theta} \right),$$
 (11)

where $Z_{-l} = \overline{Z}_l$. This expansion is analogous to a Fourier series expansion where Z_l corresponds to the Fourier coefficients. Using the complex order parameters for $l = \pm 1, \pm 2$, the Fokker-Plank equation (6) can be rewritten as

$$\frac{\partial P}{\partial t} = \frac{1}{2i} \frac{\partial}{\partial \theta} \left(\left\{ -2i\omega + K_1 [Z_{-1} \exp(i\theta) - Z_1 \exp(-i\theta)] + K_{2a} [Z_{-1}^2 \exp(2i\theta) - Z_1^2 \exp(-2i\theta)] + K_{2b} [Z_{-2} Z_1 \exp(i\theta) - Z_{-1} Z_2 \exp(-i\theta)] \right\} P \right) + D \frac{\partial^2 P}{\partial \theta^2}.$$
(12)

As mentioned, the term involving ω remains in Eq. (12). By further substituting Eq. (11) into Eq. (12), we obtain

$$\begin{aligned} \frac{dZ_1}{dt} &= (-i\omega - D)Z_1 - \frac{K_1}{2}(Z_2Z_{-1} - Z_1) \\ &- \frac{K_{2a}}{2}(Z_3Z_{-1}^2 - Z_{-1}Z_1^2) - \frac{K_{2b}}{2}(Z_1Z_{-2}Z_2 - Z_{-1}Z_2), \\ &= \frac{K_1 - 2D - 2i\omega}{2}Z_1 - \frac{K_1Z_2Z_{-1}}{2} \\ &- \frac{K_{2a}}{2}(Z_3Z_{-1}^2 - Z_{-1}Z_1^2) - \frac{K_{2b}}{2}(Z_1Z_{-2}Z_2 - Z_{-1}Z_2) \end{aligned}$$
(13)
$$\begin{aligned} \frac{dZ_2}{dt} &= (-2i\omega - 4D)Z_2 - K_1(Z_3Z_{-1} - Z_1^2) \\ &- K_{2a}(Z_4Z_{-1}^2 - Z_1^2) - K_{2b}(Z_1Z_{-2}Z_3 - Z_{-1}Z_2Z_1), \end{aligned}$$

$$\frac{\mathrm{d}Z_{l}}{\mathrm{d}t} = (-il\omega - l^{2}D)Z_{l} - \frac{lK_{1}}{2}(Z_{l+1}Z_{-1} - Z_{l-1}Z_{1})
- \frac{lK_{2a}}{2}(Z_{l+2}Z_{-1}^{2} - Z_{l-2}Z_{1}^{2})
- \frac{lK_{2b}}{2}(Z_{1}Z_{-2}Z_{l+1} - Z_{-1}Z_{2}Z_{l-1}) \quad \text{for } l \neq \pm 1, \pm 2.$$
(15)

Note that Z_{-1} and Z_{-2} obey the complex conjugate of the right-hand side of Eqs. (13) and (14), respectively.

Equation (13) implies that the state $Z_1 = 0$ bifurcates at $K_1 = 2D$; thus, we set

$$K_1 = 2D(1+\mu), (16)$$

(14)

where μ is the bifurcation parameter. We introduce $\varepsilon = \sqrt{|\mu|}$ and the scaled time $\tau = \varepsilon^2 t$. The time derivative then transformed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}.$$
 (17)

We expand the order parameter $Z_l(t)$ into $Z_{l,\nu}(t,\tau)$ as

$$Z_l(t,\tau) = \varepsilon Z_{l,1}(t,\tau) + \varepsilon^2 Z_{l,2}(t,\tau) + \cdots .$$
(18)

Substituting Eqs. (16)–(18) into the series (14)–(15), we obtain

$$\left(\frac{\partial}{\partial t} + il\omega + l^2 D\right) Z_{l,\nu} = B_{l,\nu}, \quad (l \neq \pm 1)$$
(19)

$$\left(\frac{\partial}{\partial t} \pm i\omega\right) Z_{\pm 1,\nu} = B_{\pm 1,\nu},\tag{20}$$

where

$$B_{1,1} = 0, (21)$$

$$B_{1,2} = \left(-D + \frac{K_{2b}}{2}\right) Z_{2,1} Z_{-1,1}, \qquad (22)$$

$$B_{1,3} = \left(-\frac{\partial}{\partial\tau} \pm D\right) Z_{1,1} - D(Z_{2,2}Z_{-1,1} + Z_{2,1}Z_{-1,2}) - \frac{K_{2a}}{2} (Z_{3,1}Z_{-1,1}^2 - Z_{-1,1}Z_{1,1}^2) - \frac{K_{2b}}{2} (Z_{-2,1}Z_{1,1}Z_{2,1} - Z_{2,2}Z_{-1,1} - Z_{2,1}Z_{-1,2}), \quad (23)$$

and, for $l \neq \pm 1, \pm 2$,

$$B_{l,1} = 0,$$
 (25)

$$B_{l,2} = lD(Z_{l-1,1}Z_{1,1} - Z_{l+1,1}Z_{-1,1}),$$
(26)

$$B_{l,3} = -\frac{\partial}{\partial \tau} Z_{l,1} + lD(Z_{l-1,1}Z_{1,2} + Z_{l-1,2}Z_{1,1} - Z_{l+1,1}Z_{-1,2})$$

$$- Z_{l+1,2}Z_{-1,1} - \frac{lK_{2a}}{2} (Z_{l+2,1}Z_{-1,1}^2 - Z_{l-2,1}Z_{1,1}^2)$$

$$- \frac{lK_{2b}}{2} (Z_{-2,1}Z_{1,1}Z_{l+1,1} - Z_{2,1}Z_{-1,1}Z_{l-1,1}).$$
(27)

Note that the plus-minus sign in Eq. (23) corresponds to the sign of μ .

Since $\exp(\mp i\omega t)$ is an eigenfunction of the operator $(\partial/\partial t \pm i\omega)$ on the left-hand side of Eq. (20) with an eigenvalue 0, the right-hand side has no corresponding component. We obtain the solvability condition from this fact as

$$\int_0^{2\pi/\omega} B_{1,\nu}(t,\tau) \exp(i\omega t) dt$$
$$= \int_0^{2\pi/\omega} B_{-1,\nu}(t,\tau) \exp(-i\omega t) dt = 0.$$
(28)

If $B_{1,\nu}$ are expanded into Fourier series as

$$B_{\pm 1,\nu} = \sum_{m=-\infty}^{\infty} B_{\pm 1,\nu}^{(m)} \exp(im\omega t), \qquad (29)$$

the solvability condition then reduces to

$$B_{1,\nu}^{(-1)} = B_{-1,\nu}^{(1)} = 0.$$
(30)

We solve the system of Eqs. (19) and (20). Because of Eqs. (20) and (21),

$$Z_{1,1} = \overline{Z_{-1,1}} = W(\tau) \exp(-i\omega t),$$
 (31)

where $W(\tau)$ has not yet been specified. Next, we obtain

$$Z_{3,1} = 0$$
 (32)

because of Eqs. (19) and (25) for l = 3. By substituting Eq. (32) into Eq. (24), $B_{2,2}$ reduces to

$$B_{2,2} = (2D + K_{2a})Z_{1,1}^2 \tag{33}$$

$$= (2D + K_{2a})W(\tau)^{2} \exp(-2i\omega t).$$
(34)

From Eq. (19), we have the equation for $Z_{2,2}$ as

$$\left(\frac{\partial}{\partial t} + 4D\right) Z_{2,2} = (2D + K_{2a})W(\tau)^2 \exp(-2i\omega t), \quad (35)$$

and

$$Z_{2,2} = \frac{(2D + K_{2a})W(\tau)^2 \exp(-2i\omega t)}{4D}$$
(36)

is a long-time solution to Eq. (35). In the same way as $Z_{3,1}$, we obtain

$$Z_{2,1} = 0, (37)$$

which leads to $B_{1,2} = 0$. Thus, $Z_{1,2}(t, \tau)$ is not dependent on *t*. Substituting Eqs. (32), (35), and (37) into Eq. (23) yields

$$B_{1,3} = \left[\left(-\frac{\partial}{\partial \tau} \pm D \right) W(\tau) - \frac{2D(2D - K_{2b}) - K_{2a}(2D + K_{2b})}{8D} \right]$$
$$\times |W(\tau)|^2 W(\tau) \exp(-i\omega t)$$
(38)

$$=B_{1,3}^{(-1)}\exp(-i\omega t).$$
(39)

From the solvability condition (30) for v = 3, we obtain the normalized equation in the lowest order:

$$\frac{\partial W(\tau)}{\partial \tau} = \pm DW(\tau) - \frac{2D(2D - K_{2b}) - K_{2a}(2D + K_{2b})}{8D} \times |W(\tau)|^2 W(\tau).$$
(40)

Finally, we obtain the equation with respect to t in the lowest order, which is the normal form for the Hopf bifurcation as below:

$$\frac{\mathrm{d}Z_1}{\mathrm{d}t} = \frac{\partial Z_1}{\partial t} + \varepsilon^2 \frac{\partial Z_1}{\partial \tau} \tag{41}$$

$$=\varepsilon\frac{\partial Z_{1,1}}{\partial t} + \varepsilon^2\frac{\partial Z_{1,2}}{\partial t} + \varepsilon^3\frac{\partial Z_{1,3}}{\partial t} + \varepsilon^3\frac{\partial Z_{1,1}}{\partial \tau} + O(\varepsilon^4)$$
(42)

$$=\varepsilon B_{1,1} + \varepsilon^2 B_{1,2} + \varepsilon^3 B_{1,3} + \varepsilon^3 \frac{\partial Z_{1,1}}{\partial \tau} + O(\varepsilon^4) \qquad (43)$$

$$=\varepsilon^3 \frac{\partial Z_{1,1}}{\partial \tau} + O(\varepsilon^4) \tag{44}$$

$$=\varepsilon^{3}\frac{\partial W(\tau)}{\partial \tau}\exp(-i\omega t)+O(\varepsilon^{4})$$
(45)

$$=\frac{K_1 - 2D}{2}Z_1 - g|Z_1|^2 Z_1 + O(\varepsilon^4), \tag{46}$$

where

$$g = \frac{2D(2D - K_{2b}) - K_{2a}(2D + K_{2b})}{8D}.$$
 (47)

Note that *g* is real in this particular system. The sign of *g* determines the bifurcation type; supercritical and subcritical bifurcations occur for g > 0 (or $K_{2a} < K_{2a}^*$) and g < 0 (or $K_{2a} > K_{2a}^*$), respectively, where

$$K_{2a}^* = \frac{2D(2D - K_{2b})}{2D + K_{2b}},\tag{48}$$

which assumes 2D for $K_{2b} = 0$ and decreases with increasing K_{2b} . This theoretical analysis clarifies the parameter region of the stable desynchronized state and the change in the bifurcation nature at $K_{2a} = K_{2a}^*$, which are in perfect agreement with the phase diagrams shown in Figs. 3 and 10 in Appendix C.

IV. TRANSIENT DYNAMICS: TIME EVOLUTION OF R_1

We shift our focus to transient phenomena. As we observed in Fig. 1, the synchronized state slowly decays to the asynchroous state. To elucidate the mechanism of this slow process, Eq. (41) is not adequate because it is valid only around $R_1 = 0$. We will demonstrate that under some assumptions including $K_{2b} = 0$, a dynamical equation for R_1 , which is approximately valid for large R_1 , may be obtained in



FIG. 4. Graphical representation of the potential $U(\theta, R_1)$. $R_1 = 0.6, K_1 = 0.50$.

a closed form, whereby we may determine the lifetime of the synchronized state in cases where the system transitions to the desynchronized state.

We rewrite Eq. (3) as the following graident system:

$$\dot{\theta}_m = -\frac{\partial}{\partial \theta_m} U(\theta_m, R_1) + \xi_m, \tag{49}$$

where

$$U(\theta, R_1) = -\frac{1}{2} \left(K_{2a} R_1^2 \cos 2\theta + 2K_1 R_1 \cos \theta \right).$$
 (50)

We show typical U shapes as a function of θ in Fig. 4. We focus on the case in which this function has two minima, i.e., U is double-well. Because

$$\frac{\partial}{\partial \theta} U(\theta, R_1) = K_{2a} R_1^2 \sin 2\theta + K_1 R_1 \sin \theta$$
$$= R_1 \sin \theta (2K_{2a} R_1 \cos \theta + K_1), \qquad (51)$$

the necessary and sufficient condition for the potential to be double-well is

$$\frac{K_1}{2K_{2a}R_1} < 1. (52)$$





FIG. 6. Lifetime of the synchronized states vs K_{2a} . Simulation results of Eq. (1) with $N = 10^3$ oscillators and the theoretical prediction, given by Eq. (70), are shown by colored plots and solid lines, respectively. We fix $K_1 = 0$, $K_{2b} = 0$, and D = 0.1.

Below, we assume Eq. (52). The following quantities will be needed later. The minima of the potential $U(\theta, R_1)$ are denoted as

$$U_{\min 1} := U(0, R_1) = -\frac{1}{2} \left(K_{2a} R_1^2 + 2K_1 R_1 \right), \quad (53)$$

$$U_{\min 2} := U(\pi, R_1) = -\frac{1}{2} \left(K_{2a} R_1^2 - 2K_1 R_1 \right).$$
 (54)

The maximum value U_{max} is

$$U_{\max} := U(\theta_{\max}, R_1)$$

= $-\frac{1}{2} \left(K_{2a} R_1^2 \cos 2\theta_{\max} + 2K_1 R_1 \cos \theta_{\max} \right)$
= $\frac{1}{2} \left(K_{2a} R_1^2 + \frac{K_1^2}{2K_{2a}} \right),$ (55)

where θ_{max} is defined as one of two maximum points of the potential $U(\theta)$ within the range $0 < \theta < \pi$. The potential barriers are

$$\Delta U_1 := U_{\max} - U_{\min 1} = K_{2a}R_1^2 + K_1R_1 + \frac{K_1^2}{4K_{2a}}, \quad (56)$$



FIG. 5. Comparison between the time evolutions of R_1 and the simulations of $N = 10^3$ oscillators. The black lines represent the numerical simulations of R_1 following Eq. (68), while the orange lines show the dynamics of R_1 , which are averaged over 10^3 oscillators governed by Eq. (1).



FIG. 7. Time evolution of order parameter R_1 and mean phases Θ_1 and Θ_2 , for $K_{2b} = 0.3$, $K_{2a} = 4.0$, and D = 0.1. (a) $N = 10^5$ and (b) $N = 10^3$.

$$\Delta U_2 := U_{\max} - U_{\min 2}$$

= $K_{2a}R_1^2 - K_1R_1 + \frac{K_1^2}{4K_{2a}}.$ (57)

The second derivatives of the potential at three extremal points are

$$\partial_{\theta}^{2} U(\theta_{\max}, R_{1}) = \frac{K_{1}^{2}}{2K_{2a}} - 2K_{2a}R_{1}^{2},$$
(58)

$$\partial_{\theta}^{2} U(\theta_{\min 1}, R_{1}) = \partial_{\theta}^{2} U(0) = 2K_{2a}R_{1}^{2} + K_{1}R_{1},$$
 (59)

$$\partial_{\theta}^{2} U(\theta_{\min 2}, R_{1}) = \partial_{\theta}^{2} U(\pi, R_{1}) = 2K_{2a}R_{1}^{2} - K_{1}R_{1}.$$
 (60)

We assume that the noise is sufficiently weak compared to ΔU , i.e., $D \ll \Delta U$. We also assume that R_1 evolves sufficiently slowly. Then, we can expect that the phase distribution is well approximated to

$$P(\theta, t) = \eta(t)\delta(\theta) + [1 - \eta(t)]\delta(\theta - \pi).$$
(61)

In this approximation, each oscillator takes the phase either 0 or π . By defining *H* and *H*^{*} as the states in which the phase of an oscillator has the phases 0 and π , respectively, the transition process is schematically described as

$$H \stackrel{k_+}{\underset{k_-}{\rightleftharpoons}} H^*, \tag{62}$$



FIG. 8. Typical behavior of the self-consistency equation (10) for $K_{2a} = 1.0$ and $K_{2b} = 0$ with varying K_1 .

where k_{\pm} are the transition rates given by

$$k_{+}(R_{1}) = 2 \frac{\sqrt{\left|\partial_{\theta}^{2} U(\theta_{\min 1}, R_{1})\partial_{\theta}^{2} U(\theta_{\max}, R_{1})\right|}}{2\pi} \exp\left(-\frac{\Delta U_{1}}{D}\right),$$
(63)

$$k_{-}(R_{1}) = 2 \frac{\sqrt{\left|\partial_{\theta}^{2} U(\theta_{\min 2}, R_{1})\partial_{\theta}^{2} U(\theta_{\max}, R_{1})\right|}}{2\pi} \exp\left(-\frac{\Delta U_{2}}{D}\right).$$
(64)

Substituting the obtained expressions into Eqs. (63) and (64), we obtain

$$k_{+}(R_{1}) = \frac{1}{\pi} \sqrt{\left(2K_{2a}R_{1}^{2} + K_{1}R_{1}\right)\left|2K_{2a}R_{1}^{2} - \frac{K_{1}^{2}}{2K_{2a}}\right|} \times \exp\left[-\frac{1}{D}\left(K_{2a}R_{1}^{2} + K_{1}R_{1} + \frac{K_{1}^{2}}{4K_{2a}}\right)\right], \quad (65)$$

$$k_{-}(R_{1}) = \frac{1}{\pi} \sqrt{\left(2K_{2a}R_{1}^{2} - K_{1}R_{1}\right) \left|2K_{2a}R_{1}^{2} - \frac{K_{1}^{2}}{2K_{2a}}\right|} \\ \times \exp\left[-\frac{1}{D}\left(K_{2a}R_{1}^{2} - K_{1}R_{1} + \frac{K_{1}^{2}}{4K_{2a}}\right)\right]. \quad (66)$$

We find that the time evolution of R_1 is

$$= -k_{+}(1+R_{1}) + k_{-}(1-R_{1})$$

$$= -\frac{1}{\pi} \sqrt{\left(2K_{2a}R_{1}^{2} + K_{1}R_{1}\right)\left(2K_{2a}R_{1}^{2} - \frac{K_{1}^{2}}{2K_{2a}}\right)}$$

$$\times \exp\left[-\frac{1}{D}\left(K_{2a}R_{1}^{2} + K_{1}R_{1} + \frac{K_{1}^{2}}{4K_{2a}}\right)\right](1+R_{1})$$

$$+ \frac{1}{\pi} \sqrt{\left(2K_{2a}R_{1}^{2} - K_{1}R_{1}\right)\left(2K_{2a}R_{1}^{2} - \frac{K_{1}^{2}}{2K_{2a}}\right)}$$

$$\times \exp\left[-\frac{1}{D}\left(K_{2a}R_{1}^{2} - K_{1}R_{1} + \frac{K_{1}^{2}}{4K_{2a}}\right)\right](1-R_{1}).$$

$$(67)$$

Figure 5 compares the dynamics of R_1 given by (68) to those of the simulations of $N = 10^3$ oscillators governed by Eq. (1).

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FIG. 9. Visualization of the iteration [(B1), (B2)] for $K_1 = 0.1$, $K_{2a} = 4.0$, $K_{2b} = 0.05$, and D = 0.1. The black and orange symbols present the time courses of R_1^k and R_2^k for $(R_1^0, R_2^0) = (0.75, 0.75)$ and $(R_1^0, R_2^0) = (0.1, 0.1)$, respectively.

This comparison is made in a regime where the synchronized states are stable, showing excellent agreement.

To specifically focus on the lifetime of the synchronized state, we now consider the simplified case of $K_1 = 0$, where Eq. (68) reduces to the following form:

$$\dot{R_1}(t) = -\frac{4K_{2a}R_1^3}{\pi} \exp\left(-\frac{K_{2a}R_1^2}{D}\right).$$
 (69)

Because $\hat{R}_1 < 0$ for $R_1 > 0$ and $\hat{R}_1 = 0$ for $R_1 = 0$, $R_1 = 0$ is the global attractor. However, as the term $\exp(-K_{2a}R_1^2/D)$ is vanishingly small for $R_1 \gg \sqrt{\frac{D}{K_{2a}}} \equiv R^*$, the relaxation to $R_1 =$ 0 is extremely slow if $R_1(0) \equiv \hat{R} \gg R^*$. We define the lifetime τ of the synchronized state as the time for which R_1 varies from \hat{R} to R^* . Because $\tau = \int_0^{\tau} dt = \int_{\hat{R}}^{R^*} \frac{dt}{dR_1} dR_1$, we obtain

$$\tau = \int_{R^*}^{\hat{R}} \frac{\pi \exp\left(\frac{K_{2a}R_1^2}{D}\right)}{4K_{2a}R_1^3} \, \mathrm{d}R_1. \tag{70}$$

This integral can only be computed numerically. Because the evolution of $R_1(t)$ is very slow until R_1 reaches R^* , a rough estimate of τ can be given by setting $R_1(t) = \hat{R}$ in Eq. (70),



FIG. 10. Phase diagram in the (K_1, K_{2a}) plane for $K_{2b} = 0.05$ and D = 0.1. The color scale describes the R_1 value of the stable synchronized state. The hatched region denotes the region in which the desynchronized state is stable.

giving rise to

$$au \sim \exp\left(\frac{K_{2a}\hat{R}^2}{D}\right),$$
 (71)

where the coefficient, including the factor $\frac{\hat{R}-R^*}{\hat{R}^3}$, is omitted. This estimation indicates that τ approximately exponentially increases with K_{2a} . To verify our theory, in Fig. 6, we compare our theoretical estimations, given by Eqs. (70) and (71), to the lifetime obtained from the direct simulations of Eq. (1). The lifetime is given as the time at which R_1 passes R^* for the first time. Note that $\hat{R} = |2\eta(0) - 1|$, where $\eta(0)$ is the parameter for the initial distribution. We observe that Eq. (70) is in reasonable agreement with the simulation data. We also observe that the lifetime indeed increases approximately exponentially with K_{2a} , as predicted by Eq. (71), supporting the robustness of this phenomenon (see Appendix D).

V. CONCLUSION

In this study, we explored a large population of noisy oscillators with one- and two-simplicial interactions. We demonstrated that dynamical noise, regardless of its strength, erodes the synchronized states when the one-simplex and type-b two-simplex interactions are absent or sufficiently weak. However, the lifetime of the synchronized state is prolonged, increasing exponentially with the strength of the type-a two-simplex interaction. Note that the characteristic time scales of individual units, given by the inverse of interaction and the noise strength, are small in our setting. For example, the time scale of the diffusion process is $\frac{1}{\sqrt{D}}$, which is approximately 3. Compared to them, the erosion process is extremely slow. Thus, the system can be considered synchronous or asynchronous depending on the time scale of the observation. We also emphasize that the erosion is robust, which is observed in the presence of different types of interactions and frequency heterogeneity. The proposed theory, which can handle mixed systems of one-simplex and two types of two-simplex interactions, clarified the bifurcation structure and transient dynamics.



FIG. 11. Lifetime of synchronization vs K_{2a} for (a) $K_1 = 0.1$ and $K_{2b} = 0$, and (b) $K_1 = 0$ and $K_{2b} = 0.1$. Results of simulations for $N = 10^3$ and D = 0.1.

APPENDIX A: TIME EVOLUTION OF Θ_1 AND Θ_2

Figure 7, which illustrates the results of direct simulations of Eq. (1), shows that the trajectories of $2\Theta_1$ are the same as those of Θ_2 in synchronized states. Moreover, Θ_1 and Θ_2 evolve more slowly as the number of oscillators increases. Therefore, we expect that $\dot{\Theta}_l \rightarrow 0$ as $N \rightarrow \infty$.

APPENDIX B: NUMERICAL ANALYSIS OF SELF-CONSISTENCY EQUATION

In this Appendix, we briefly illustrate a numerical analysis of the self-consistency equation, given by Eq. (10). For $K_{2b} = 0$, we need to solve Eq. (10) only for l = 1 because R_2 is not involved. Figure 8 shows $S_1(R_{1s})$ vs R_{1s} and R_{1s} vs R_{1s} , and its concurrent points can easily be identified using, e.g., a bisection method.

For $K_{2b} \neq 0$, we use the following iterative method:

$$R_1^{k+1} = S_1(R_1^k, R_2^k), \tag{B1}$$

$$R_2^{k+1} = S_2(R_1^k, R_2^k).$$
(B2)

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We expect that (R_{1s}, R_{2s}) corresponding to a stable steady state of Eq. (6) is obtained for an appropriate initial condition. In Fig. 9, we plot the values of R_l^k at each step of iteration in the bistable region. We observe that either the desynchronized or synchronized state is obtained depending on the initial condition.

APPENDIX C: PHASE DIAGRAM IN THE (K_1, K_{2a}) PLANE FOR NONVANISHING K_{2b}

Figure 10 is a phase diagram in the (K_1, K_{2a}) plane for $K_{2b} = 0.05$. The point in Fig. 10 marks the bifurcation type switch at $K_1 = 2D$, as described by Eq. (48).

APPENDIX D: LIFETIME OF SYNCHRONIZATION FOR $K_1 > 0$ OR $K_{2b} > 0$

In Fig. 11, we plot the lifetime of synchronization when the one-simplex or type-b interaction is present ($K_1 = 0.1$ or $K_{2b} = 0.1$). In this case, no synchronized states are stable for any K_{2a} . We find that the lifetime has an exponential dependence on K_{2a} as well as when $K_1 = 0$ and $K_{2b} = 0$.

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